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On the Stochastic Dynamics of Ising Models

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With the help of recent results in the mathematical theory of master equations, we present a rigorous derivation of the stochastic Glauber dynamics of Ising models from Hamiltonian quantum mechanics. A thermal bath is explicitly constructed and, as an illustration, the dynamics of the Ising–Weiss model is analyzed in the thermodynamic limit. We thus obtain an example of a nonequilibrium statistical mechanical system for which a link without mathematical gap can be established from microscopic quantum mechanics to a macroscopic irreversible thermodynamic process.

KEY WORDS: Stochastic dynamics; Ising model; thermal bath; master equation; Markov process; nonequilibrium thermodynamics.

1. INTRODUCTION

In 1963, Glauber proposed a stochastic Ising model in which the spins change their state randomly with time according to a continuous Markov process.[1] The underlying physical picture is that the spin system, considered as open, interacts with an external heat bath with which it can exchange energy and which causes the spins to flip randomly at a given rate.

The success of the model lies in the fact that it can be used as a starting point for various interesting investigations. First, it provides a simple example of a dissipative system whose time-dependent behavior can be precisely analyzed in some cases. Second, it furnishes an efficient tool to investigate some dynamical aspects of the Ising model near the critical point analytically as well as by numerical calculations (see, for instance, Refs. 2 and 3 and the references quoted there). The stochastic Ising model has also been considered in connection with the definition of metastable states in a mean field treatment of the ferromagnet[4] and, more recently, in the case of short-range forces.[5] Moreover, it constitutes a subject of mathematical

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interest by itself, in the framework of the theory of Markov processes and Gibbs states on lattices (see, for instance, Refs. 6 and 7 and the references quoted there).

However, there are some basic questions concerning the physical foundations of the model which have not yet been fully elucidated. They are the following:

(a) Is it possible to derive the stochastic dynamics of Ising models from the principles of Hamiltonian mechanics?

(b) How do the effects of the coupling with the external agency manifest themselves on the dynamics of spins?

A justification of the stochastic dynamics and an answer to point (a) has been attempted in Ref. 8 on the basis of the Bloch and Wangsness relaxation theory,[9,10] but this study is unsatisfactory insofar as the mathematical validity of the latter theory is not established and the thermal bath is not explicitly constructed.

The purpose of this work is to present a rigorous derivation of the stochastic dynamics from the Hamiltonian mechanics of the coupled systems in the so-called weak coupling limit. This is done with the help of recent mathematical progress in the theory of master equations,[11] which we briefly recall in Appendix A.

A finite-dimensional quantum system Σ_1 is coupled with a heat bath Σ_2 of infinite size. At time $t = 0$ the states of the two systems are assumed to be uncorrelated: The state of Σ_1 is arbitrarily prepared, whereas the bath is in thermal equilibrium. The interaction is switched on, and one follows the evolution of the part of the total state reduced to the system of interest by taking a partial trace on the degrees of freedom of the bath. The reduced state obeys a general integrodifferential equation (generalized master equation), but not much can be said on its solutions, except in a favorable limiting situation.

This limiting case is obtained by a scaling of the time t, setting $\tau = \lambda^2 t$ and letting $\lambda \to 0$, $t \to \infty$, with τ fixed, where λ measures the strength of the coupling between Σ_1 and the bath. One can say heuristically that in this idealized situation (first considered by Van Hove[12]) one takes into account the nonvanishing cumulative effects on Σ_1 as $t \to \infty$ of its very weak interaction with an extensive external system.

The point is that the generalized master equation is mathematically tractable in the weak coupling limit, where it can be shown to reduce to the usual Markovian (Pauli type) master equation.

In Section 2, we formulate the model that we want to consider. We do not aim to achieve maximal generality, but on the contrary suppress any irrelevant complexity for the sake of mathematical transparency. The bath

Σ_2 will consist of quasifree assemblies of Fermi particles or elementary excitations with a given energy spectrum[2] (for instance, the free nonrelativistic Fermi gas). A finite quantum lattice spin system or more precisely its classical part, i.e., the z component of the spin angular momentum, constitutes the object of interest Σ_1. The two systems will be able to exchange energy via an interaction which we choose to be linear both in the individual spin operators and the Fermi fields. We show then that the generator of the Markov process on the classical states of the spins obtained in the weak coupling limit coincides with the spin flip process postulated in the stochastic Ising dynamics of Glauber. Moreover, an important feature which emerges from this treatment is that the usually arbitrary parameters occurring in the transition probabilities are now completely specified by the nature of the bath and of the coupling. In particular it is well known that the thermal state of a quasifree field is uniquely determined by its two-point correlation function. It will be seen that the rate at which transitions between spin states take place is essentially given in terms of the Fourier transform of this correlation function.

In order to illustrate these points in an explicitly solvable model, we study in Section 3 the dynamics of the open Ising–Weiss model. We derive an evolution law for the probability distribution of a macroscopic observable, the magnetization density. The link between microscopic and macroscopic evolution is established by considering a certain class of macroscopic states for which we obtain an irreversible equation of motion in the thermodynamic limit. The latter equation, which is similar to those discussed in Ref. 2, is solved without difficulty. The bifurcation which occurs at the critical temperature is discussed, as well as the influence of the bath on the relaxation properties. The effects due to the finite number N of spins can also be taken into account. In particular, to order $1/N$, the magnetization obeys a Fokker–Planck equation. The fluctuations are normal in the neighborhood of a stable equilibrium point except when the temperature is critical, in which case they are of order $N^{-1/4}$ instead of the usual $O(N^{-1/2})$. Finally, Appendix B is devoted to the study of the mathematical structure of the macroscopic evolution law. The point is that in each phase the full evolution is asymptotic, as the time goes to infinity, to the motion obtained by linearizing the equation around the equilibrium point. The full motion and its linearization are linked by an asymptotic relation which is similar to that occurring in scattering theory, and the technique may prove to be useful in cases where the equation of motion cannot be solved explicitly.

Before concluding this introduction, we point out that what this model gives us is an example of a nonequilibrium statistical mechanical system for

[2] The Fermi rather than the Bose statistics are chosen for mathematical convenience.

which a link without mathematical gap can be established from microscopic quantum mechanics to a macroscopic irreversible thermodynamic process. For this reason, we found it worthwhile to expose it to the full extent. It is important in particular to notice the involved limits and the order in which they occur. First of all the thermodynamic limit is taken on the bath, keeping the spin system finite (technically, this is done in Section 2 by considering from the beginning the bath as an infinite quantum mechanical system in a Kubo–Martin–Schwinger state). Then the dynamics of the spins is studied in the weak coupling limit and finally the thermodynamic limit is also taken on the spins for a suitable class of nonequilibrium macroscopic states (Section 3). This order is not irrelevant; in particular, since both systems are eventually macroscopic, we ensure that the Fermi fields really play their role of a thermal bath by first taking their thermodynamic limit.

The model is of course subject to criticism related to its mean field nature (i.e., size-dependent forces), and one may also question the interpretation of the limiting procedures. However, it has the merit of presenting a scheme in which every step can be exposed and discussed with full clarity.

2. DERIVATION OF THE STOCHASTIC DYNAMICS OF ISING MODELS

We start by specifying the microscopic models of the spin system and of the heat bath.

2.1. The Spin System Σ_1

The spin system Σ_1 consists of a finite quantum lattice Λ of spins $\frac{1}{2}$ with N lattice sites. We denote by

$$\sigma_j^x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_j^y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_j^z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},$$

$$\sigma_j^{\pm} = \tfrac{1}{2}(\sigma_j^x \pm i\sigma_j^y)$$

the spin operators attached to the lattice site j and acting on $\mathbb{C}_j^2$. An Ising-type Hamiltonian is given on $\mathscr{H}_1 = \prod_{j\in\Lambda} \mathbb{C}_j^2$ by

$$H_1 = \tfrac{1}{2} \sum_{i,j\in\Lambda} J_{ij}\sigma_i^z\sigma_j^z$$

The coupling constants $J_{ij} = J_{ji}$ are real, with $J_{ii} = 0$, and they are not subjected here to any other requirement.

We define the classical part of Σ_1 as follows. Let $\mathscr{A}^z$ be the algebra generated by the operators $\sigma_j^z, j \in \Lambda$. If μ is any density matrix on $\mathscr{H}_1$, we call the classical part of the state the restriction of μ to the algebra $\mathscr{A}^z$. Let ω

be a configuration of the Ising model, where ω is a function from Λ into $\{-1, 1\}$, $\omega(j) = \pm 1$ giving the value of the z component of the spin at the site j. To each configuration ω we associate the eigenvector $\chi_\omega \in \mathscr{H}_1$, which is common to all σ_j^z, with $\sigma_j^z\chi_\omega = \omega(j)\chi_\omega$. Then $\mathscr{A}^z$ is identified with the set of functions $A(\omega)$ on configuration space by $A\chi_\omega = A(\omega)\chi_\omega$, and the diagonal elements $(\chi_\omega, \mu\chi_\omega) = \mu(\omega)$ of μ define a probability distribution on configuration space with

$$\mathrm{Tr}_{\mathscr{H}_1} \mu A = \sum_\omega \mu(\omega)A(\omega)$$

Moreover, we say that μ is classical if μ has no off-diagonal matrix elements, i.e., $(\chi_\omega, \mu\chi_{\omega'}) = 0$ when $\omega' \neq \omega$.

2.2. The Thermal Bath Σ_2

The bath consists of quasifree Fermi fields at thermal equilibrium. We denote by $\hbar$ the one-particle Hilbert space and by $\varphi(f)$ the Fermi field operator smeared with a function f in $\hbar$. The state of a quasifree Fermi field at temperature $T = (k\beta)^{-1}$ (k is the Boltzmann constant) is completely determined by its two-point correlation function[13]

$$\begin{aligned}\langle\varphi(f)\varphi(g)\rangle_\beta = (f, g) + ([\exp \beta h + 1]^{-1}g, f)\\ - ([\exp \beta h + 1]^{-1}f, g), \qquad f, g \in \hbar \qquad (1)\end{aligned}$$

where (f, g) is the scalar product in $\hbar$ and h is the one-particle Hamiltonian.

The Fermi operators $\varphi(f)$, $f \in \hbar$, satisfy the anticommutation relations on the Hilbert space $\mathscr{H}$ of the representation defined by (1). The space $\mathscr{H}$ is constructed from a vector Ω cyclic with respect to the algebra generated by the field operators, with the property

$$(\Omega, \varphi(f)\Omega) = 0, \qquad (\Omega, \varphi(f)\varphi(g)\Omega) = \langle\varphi(f)\varphi(g)\rangle_\beta \qquad (2)$$

The time evolution of the field is implemented on $\mathscr{H}$ by an evolution operator $\exp(-iHt)$ such that

$$\varphi(\exp(-iht)f) = \exp(-iHt)\,\varphi(f)\exp(iHt)$$

and Ω is left-invariant

$$\exp(-iHt)\Omega = \Omega \qquad (3)$$

For a given $f \in \hbar$ we denote by $C(t) = \langle\varphi(\exp(-iht)f)\varphi(f)\rangle_\beta$ the two-point, time-dependent correlation function, and by $\tilde{C}(x) = \int dt \exp(ixt)C(t)$ its Fourier transform.[3]

[3] In the distribution sense if $C(t)$ is not integrable.

Then it follows from (1) that $\tilde{C}(x) \geqslant 0$ and $\tilde{C}(x)$ satisfies the Kubo–Martin–Schwinger relation:

$$\tilde{C}(-x) = \exp(\beta x)\,\tilde{C}(x) \tag{4}$$

For instance, we have in the free nonrelativistic case

$$\mathscr{h} = \mathscr{L}^2(R^3), \qquad (\exp(-iht)f)(\mathbf{p}) = \exp\left(-i\frac{p^2}{2m}t\right)f(\mathbf{p})$$

$$C(t) = \int d^3p\,|f(\mathbf{p})|^2\,\frac{\exp[i(p^2/2m)(t-i\beta)] + \exp[-i(p^2/2m)t]}{\exp(\beta(p^2/2m)) + 1} \tag{5}$$

We construct now our thermal bath as follows. To each lattice site $j \in \Lambda$ we associate an identical and independent copy of the above-described Fermi system with Hilbert space $\mathscr{K}_j$, cyclic vector Ω_j, field operator $\varphi_j(f)$, and Hamiltonian H_j. Therefore Σ_2 consist of N identical uncoupled Fermi systems on $\mathscr{H}_2 = \prod_{j\in\Lambda}\mathscr{K}_j$ with total Hamiltonian $H_2 = \sum_{j\in\Lambda} H_j$.

Finally, we couple each spin with its own bath linearly in the spin and the field operators, setting

$$V = \sum_{j\in\Lambda} \sigma_j^x \varphi_j(f) \tag{6}$$

where the coupling function f is a fixed element in $\mathscr{h}$.

This concludes the description of the model.

2.3. The Stochastic Dynamics

We apply to the model the formalism of generalized master equations. If we choose the initial state of Σ_2 to be equilibrium state $\prod_{j\in\Lambda}\Omega_j$ of the bath, we remark that we are in the mathematical setting of Ref. 11 (see Appendix A). $\mathscr{H}_1$ is finite dimensional and V is bounded on $\mathscr{H}_1 \otimes \mathscr{H}_2$ since the Fermi field operators are bounded on $\mathscr{K}$. Moreover, due to (3) and (2), the properties (ii) and (iii) of Appendix A are satisfied.

It is not an elementary task to check that the weak coupling limit theorem applies. This is indeed the case if there is a sufficiently fast decrease of the time correlation functions of the bath.

Proposition 1. Suppose that the coupling function f in (6) is such that $|C(t)| \leqslant a/(1+|t|)^{1+\epsilon}$ for some $a > 0$ and $\epsilon > 0$; then the weak coupling limit theorem holds true, and the generator G of the evolution of the classical part of the spin system is given by formula (A.9).

For the proof, we refer the reader to the Section 2 of Ref. 11. A slight difference from the case treated in Ref. 11 is that our bath is made of a finite number of Fermi fields instead of only one. The integrability of the time correlation functions of the bath is essential for the proof. This property depends in turn on the behavior at $\mathbf{p} = 0$ of the energy spectrum of the excitations in the bath. It is verified, for instance, for the p^2 law, since it is well known that $C(t)$ given by (5) satisfies the estimate $C(t) = O(t^{-3/2})$ for a large class of $f \in \mathscr{L}^2(R^3)$ (spreading of the wave packet).

Proposition 2. Suppose that the initial state μ of the spin system is classical. Then:

(a) The state μ_τ is classical for all $\tau \geqslant 0$.

(b) The semigroup defines a Markov process on the set of configurations ω with transition matrix given by

$$(G\mu)(\omega) = \sum_{j\in\Lambda} \gamma(m_j(\omega))\{[1 - \omega(j) \tanh \beta m_j(\omega)]\mu(\omega_j) - [1 + \omega(j) \tanh \beta m_j(\omega)]\mu(\omega)\} \tag{7}$$

with

$$\gamma(x) = \tfrac{1}{2}[\tilde{C}(2x) + \tilde{C}(-2x)] = \gamma(-x) \geqslant 0$$

$$m_j(\omega) = \sum_{k\in\Lambda} J_{ik}\omega(k); \qquad \omega_j(k) = \begin{cases} \omega(k), & k \neq j \\ -\omega(k), & k = j \end{cases}$$

(c) The canonical distribution $\mu_\beta(\omega) = \{\exp[-\beta H_1(\omega)]\}/\sum_\omega \exp[-\beta H_1(\omega)]$ is a stationary state. If $\mathbf{C}(x)$ is strictly positive, μ_β is the unique stationary state and every state converges to it as $\tau \to \infty$.

Proof. To verify (a) and (b) one has to write G explicitly in terms of the interaction (6). We notice first that in virtue of the properties (ii) and (iii), (A.5) gives simply

$$K(0, s) = P\mathscr{V}U_s^0\mathscr{V}P \tag{8}$$

Inserting in (8) the definition of the operators $\mathscr{V}$ and P, one gets

$$\begin{aligned} G\mu &= -\int_0^\infty ds\, P\left[V, U_s^0\left(\left[V, \mu \otimes \prod_{j\in\Lambda} \Omega_j\right]\right)\right] \\ &= -\int_0^\infty ds \sum_{ij\in\Lambda} [\sigma_i^x\sigma_j^x(s)\mu C_{ij}(-s) - \sigma_i^x\mu\sigma_j^x(s)C_{ji}(s) \\ &\quad - \sigma_j^x(s)\mu\sigma_i^x C_{ij}(-s) + \mu\sigma_j^x(s)\sigma_i C_{ji}(s)] \\ &= -\int_0^\infty ds \sum_{j\in\Lambda} [\sigma_j^x\sigma_j^x(s)\mu C(-s) - \sigma_j^x\mu\sigma_j^x(s)C(s) \\ &\quad - \sigma_j^x(s)\mu\sigma_j^x C(-s) + \mu\sigma_j^x(s)\sigma_j^x C(s)] \end{aligned} \tag{9}$$

In obtaining (9) use has been made of the stationarity of μ and $\prod_{j\in\Lambda}\Omega_j$ under the free evolution and of the fact that Fermi fields attached to different lattice sites are uncorrelated:

$$C_{ij}(s) = \langle \varphi_i[\exp(-ihs)f]\varphi_j(f)\rangle = \delta_{ij}C(s)$$

The time evolution of the spin operators in (9) is given by

$$\begin{aligned}\sigma_j{}^x(s) &= \exp(-iH_1 s)\sigma_j{}^x \exp(iH_1 s)\\ &= \exp(-2im_j s)\sigma_j{}^+ + \exp(2im_j s)\sigma_j{}^- \end{aligned} \tag{10}$$

with $m_j = \sum_{k\in\Lambda} J_{jk}\sigma_k{}^z$.

Equations (9) and (10) show that if μ is classical, $G\mu$ has no off-diagonal elements. Indeed, terms in (9) that are bilinear in $\{\sigma_j{}^+, \sigma_j{}^+\}$ or $\{\sigma_j{}^-, \sigma_j{}^-\}$ vanish and terms that are bilinear in $\{\sigma_j{}^+, \sigma_j{}^-\}$ or $\{\sigma_j{}^-, \sigma_j{}^+\}$ are diagonal on the states χ_ω. Then the same property holds for $G^n\mu$, $n = 2, 3,\ldots$, and hence for $\mu_\tau = \sum_{n=0}^{\infty} (\tau^n/n!)G^n\mu$, showing that (a) is true.

Part (b) is obtained by calculating the diagonal elements $(\chi_\omega, G\mu\chi_\omega)$. We have

$$\begin{aligned}(\chi_\omega, \sigma_j{}^x\sigma_j{}^x(s)\mu\chi_\omega) &= \{\exp[2im_j(\omega)s](1 + \omega(j))/2\\ &\quad + \exp[-2im_j(\omega)s](1 - \omega(j))/2\}\mu(\omega)\\ (\chi_\omega, \sigma_j{}^x\mu\sigma_j{}^x(s)\chi_\omega) &= \{\exp[2im_j(\omega)s](1 + \omega(j))/2\\ &\quad + \exp[-2im_j(\omega)s](1 - \omega(j))/2\}\mu(\omega_j)\end{aligned}$$

and similar expressions for the other terms in (9). Inserting this in (9) and using the Kubo–Martin–Schwinger condition (4) for the Fourier transform $\tilde{C}(x)$ of the correlation function $C(t)$, one gets (7). The fact that the canonical distribution is stationary is seen by direct calculation.

If $\tilde{C}(x) > 0$ for all x, $\gamma(m_j(\omega))$ is strictly positive for all configurations ω and all $j \in \Lambda$. Therefore all pairs of configurations $\{\omega, \omega_j\}$ differing by one spin flip are connected by the nonvanishing matrix element $\gamma(m_j(\omega)) \times [1 - \omega(j)\tanh \beta m_j(\omega)]$. From this, we conclude that all pairs of configurations are connected by a sequence of nonvanishing matrix elements; thus the process is irreducible and part (c) follows.

The master equation (7) has essentially the same structure as that which can be found in the literature. However, it is expressed here in terms of the rescaled evolution parameter τ, and the transition rate of spin flips is not a constant independent of the configurations, but it is determined by the function $\tilde{C}(x)$. We should add that the validity of this analysis is not limited to the interaction V that we have chosen. By modifying the spin operator part in V, one can generate a variety of processes, all of them admitting the canonical distribution as stationary state. In particular it is clear that polynomial interactions in the single spin operators will lead to multi-spin-flip Markov processes. The weak coupling limit theory can also be extended to the case where polynomials in the Fermi fields are considered.[11]

3. DYNAMICS OF THE OPEN ISING–WEISS MODEL

In the Ising–Weiss model every spin interacts equally with every other spin through size-dependent forces. The model is defined by setting $J_{ij} = -J/N$, $i \neq j$, $J > 0$.[14]

We are interested in the evolution equation for the probability distribution of the magnetization density $\alpha^N(\omega) = (1/N) \sum_{j\in\Lambda} \omega(j)$ in the thermodynamic limit. In order to formulate precisely the passage from the microscopic to the macroscopic description, we introduce the following class of states μ^N.[4] Consider first the set of observables $A(\omega)$ on phase space which are functions only of the macroscopic observable α^N:

$$A(\omega) = f(\alpha^N(\omega))$$

where $f(x)$ belongs to the class C_0 of real-valued continuous functions on R with compact support. We say that the sequence μ^N of states is macroscopic if

$$\lim_{N\to\infty} \mu^N(f) = \lim_{N\to\infty} \sum_{\omega} \mu^N(\omega) f(\alpha^N(\omega)) = \mu(f)$$

exists for all $f \in C_0$. The limit $\mu(f)$ is then clearly a bounded, positive linear functional on C_0. By the Riesz representation theorem, it is given by a probability measure $d\mu(\alpha)$ on R such that $\mu(f) = \int d\mu(\alpha) f(\alpha)$. Here μ is the probability measure of the macroscopic observable associated with the states μ^N.

The next proposition shows that in the class of macroscopic states of the Ising–Weiss model, the microscopic evolution (7) induces an evolution equation for the macroscopic probability measures μ. Furthermore, the evolution equation for μ is generated by a simple nonlinear differential equation for the magnetization density α.

Proposition 3. Let μ^N be a macroscopic sequence of states with associated probability measure μ on R. Then:

(a) $\lim_{N\to\infty} (G^N\mu^N)(f) = \mu(hf')$ for all f belonging to the set $C_0{}^1$ of continuously differentiable functions, with $f'(\alpha) = (d/d\alpha)f(\alpha)$ and $h(\alpha) = 2\gamma(J\alpha)(\tanh \beta J\alpha - \alpha)$.

[4] From now on we indicate the volume dependence by the index N.

(b) The evolution equation for the probability measure

$$(d/d\tau)\mu_\tau(f) = \mu_\tau(hf') \tag{11}$$

is solved by

$$\mu_\tau(f) = \mu(f_\tau), \qquad f_\tau(\alpha) = f(\alpha_\tau)$$

where α_τ is the solution of the differential equation

$$d\alpha_\tau/d\tau = h(\alpha_\tau) \tag{12}$$

with initial condition α.

Proof. We consider first the action of G^N on local observables A_Δ depending only on the variables $\omega(j)$ for $j \in \Delta \subseteq \Lambda$. One has from (7) for $\Delta \subseteq \Lambda$

$$\begin{aligned}(G^N\mu^N)(A_\Delta) &= \sum_\omega (G^N\mu^N)(\omega)A_\Delta(\omega)\\ &= \sum_\omega \sum_{j\in\Delta} \gamma(m_j(\omega))[1 + \omega(j)\tanh \beta m_j(\omega)]\\ &\quad \times [A_\Delta(\omega_j) - A_\Delta(\omega)]\mu^N(\omega)\end{aligned} \tag{13}$$

To get (13), we changed the dummy summation variable ω into ω_j and used $\omega_j(j) = -\omega(j)$, $m_j(\omega) = m_j(\omega_j)$ (since $J_{jj} = 0$).

Inserting $m_j(\omega) = -J[\alpha^N(\omega) - \omega(j)/N]$, $\alpha^N(\omega_j) = \alpha^N(\omega) - 2\omega(j)/N$, and $A_\Delta(\omega) = f(\alpha^N(\omega))$ in (13), we get

$$\begin{aligned}(G^N\mu^N)(f) &= \sum_\omega \sum_j^N \gamma\left(J\left[\alpha^N(\omega) - \frac{\omega(j)}{N}\right]\right)\\ &\quad \times \left(1 - \omega(j)\tanh \beta J\left[\alpha^N(\omega) - \frac{\omega(j)}{N}\right]\right)\\ &\quad \times \left[f\left(\alpha^N(\omega) - \frac{2\omega(j)}{N}\right) - f(\alpha^N(\omega))\right]\mu^N(\omega)\end{aligned}$$

Since f is continuously differentiable, we can write

$$f\left(\alpha^N(\omega) - \frac{2\omega(j)}{N}\right) - f(\alpha^N(\omega)) = -\frac{2\omega(j)}{N} f'(\alpha^N(\omega)) + O\left(\frac{1}{N}\right)$$

Since $\gamma(x)$ is the Fourier transform of a $\mathscr{L}^1$ function, it is uniformly continuous as well as $\tanh x$ and we have also

$$\begin{aligned}&\gamma\left(J\left[\alpha^N(\omega) - \frac{\omega(j)}{N}\right]\right)\left(1 - \omega(j)\tanh \beta J\left[\alpha^N(\omega) - \frac{\omega(j)}{N}\right]\right)\\ &\quad = \gamma(J\alpha^N(\omega))[1 - \omega(j)\tanh \beta J\alpha^N(\omega)] + O(1)\end{aligned}$$

Using the fact that the functions $\gamma(x)$, $\tanh x$, and $f'(x)$ are bounded, we deduce that

$$(G^N\mu^N)(f) = \mu^N(hf') + O(1/N)\sum_{j}^{N}\sum_{\omega}\mu^N(\omega) = \mu^N(hf') + O(1)$$

from which (a) follows. For part (b) one proceeds to a direct verification.

Let us discuss the main features of the solution of (12). Then the properties of the evolution of probability measures follow immediately from those of the one-dimensional flow generated by (12).[5]

1. $\gamma(x)$ does not vanish and is a slowly varying function. The zeros of $h(\alpha)$ are then those of $\tanh \beta J\alpha - \alpha$.

For $T > T_c$,[6] $h(\alpha)$ has a unique zero $\alpha = 0$ at the origin. It is a stable attracting point since the linearization $h^L(\alpha)$ of $h(\alpha)$ in the neighborhood of $\alpha = 0$ gives $h^L(\alpha) = -c\alpha$, with $c = 2\gamma(0)(1 - \beta J) > 0$. At $T = T_c$, one has $c = 0$ and there is a bifurcation. For $T < T_c$, two new attracting stationary points occur at $\alpha = m_+$ and $\alpha = m_-$ ($m_+ = -m_- > 0$), the origin being now an unstable stationary point. The linearized part $h^L(\alpha)$ of $h(\alpha)$ in the neighborhood of a stable equilibrium point is now $h^L(\alpha) = -c(\alpha - m)$, with

$$c = 2\gamma(Jm)[1 - \beta J(\cosh \beta Jm)^{-2}] > 0 \tag{14}$$

The domain of attraction of m_+ (resp. m_-) is the set of all positive initial conditions $\alpha > 0$ (resp. $\alpha < 0$) and the approach to equilibrium is exponential:

$$\alpha_\tau^L = (\alpha - m)\exp(-c\tau) + m$$

The mathematical sense in which the linearized motion α_τ^L approximates the full solution is made precise in Appendix B.

At $T = T_c$, the relaxation time c^{-1} diverges as $|T - T_c|^{-1}$ and the $\tanh \alpha - \alpha$ has a triple zero, giving rise to the critical slowing down $\alpha_\tau \simeq [\alpha^{-2} + \frac{4}{3}\gamma(0)\tau]^{-1/2}$.

2. We investigate the influence of the bath on the relaxation.

(a) $\gamma(J\alpha)$ has a sharp minimum at $\alpha = \alpha_1$, with $0 < \gamma(J\alpha_1) \ll 1$. Initial conditions chosen in a neighborhood of α_1 are trapped in this region for a long time, giving rise to a metastable situation.

(b) $\gamma(J\alpha)$ itself has a zero at $\alpha = \alpha_1 > 0$, which is a new stationary point. If $\gamma(x)$ is twice differentiable, this zero is necessarily double since $\gamma(x) \geqslant 0$. Then α_1 attracts initial conditions with $\alpha \geqslant \alpha_1$ and repulses initial conditions with $0 < \alpha < \alpha_1$ (for $T > T_c$).

[5] A detailed discussion can be found in Ref. 15.

[6] T_c is defined by $\beta J = (kT_c)^{-1}J = 1$.

A case of interest occurs if α_1 coincides with m_+ (in the case $T > T_c$). Then $h(\alpha)$ has a zero of order three at m_+ and this causes a considerably slower approach to equilibrium of the type

$$\alpha_\tau \simeq [(\alpha - m_+)^{-2} + \text{const} \times \tau]^{-1/2} + m_+$$

Such a slowing is comparable to that which one observes at the critical temperature, although here it is an effect due to the bath. The occurrence of these various situations is of course linked to the structure of $\tilde{C}(x)$, which depends in turn on coupling and on the energy spectrum of the bath.

3. If a constant external magnetic field B is applied, the only modification is to write $\tanh \beta J(\alpha + B)$ in $h(\alpha)$. For small B and T, $h(\alpha)$ still has two zeros [assuming $\gamma(x)$ strictly positive]. One of them is a stable equilibrium point with the direction of the magnetization opposed to that of the field. Such a phenomenon has been found in Ref. 4, where it is indicated that at finite volumes the relaxation time of the state increases exponentially with the volume. When B becomes larger ($B > B_c$) this equilibrium state disappears, to give rise to a region of metastability as in case 2(a).

4. In order to take into account the fluctuations that occur when the volume of the spin system is not strictly infinite, we calculate the first correction of order $1/N$ to the infinite-volume limit generator $\lim_{N\to\infty} G^N$ of Proposition 3(a). We find the equation

$$\frac{d}{d\tau} \mu_\tau(f) = \mu_\tau\left(hf' + \frac{1}{N} Kf''\right) \tag{15}$$

with $K(\alpha) = 2\gamma(J\alpha)(1 - \alpha \tanh \beta J\alpha)$. [Strictly speaking, h should also be modified to the order $1/N$ but we are only interested in retaining the effects of the diffusion term $(1/N)Kf''$.] For $T \neq T_c$ we again make the linear approximation in the neighborhood of a stable equilibrium point m:

$$h^L(\alpha) = -c(\alpha - m), \qquad (1/N)K^L(\alpha) = [2\gamma(Jm)/N](1 - m^2) = D$$

where c is given by (14).

If $d\mu(\alpha) = p(\alpha)\, d\alpha$ has a probability distribution $p(\alpha)$, the equation for $p_\tau(\alpha)$ corresponding to (15) is the Fokker–Planck equation:

$$\frac{\partial}{\partial\tau} p_\tau(\alpha) = c \frac{\partial}{\partial\alpha} [(\alpha - m)p_\tau(\alpha)] + D \frac{\partial^2}{\partial\alpha^2} p_\tau(\alpha)$$

with diffusion coefficient D.[7]

Hence the stationary distribution is Gaussian and the magnetization has fluctuations $(D/4c)^{1/2} = O(N^{-1/2})$.

Since $m^2 \to 1$ as $T \to 0$, D decreases with the temperature. If the temperature is critical, the behavior $h(\alpha) \simeq -\frac{2}{3}\gamma(0)\alpha^3$ leads to a stationary

[7] In this approximate calculation, we have neglected the limitation $|\alpha| \leqslant 1$.

probability distribution of the form const × $\exp[(-N/12)\alpha^4]$. As expected, the associated fluctuations are now of larger order, $O(N^{-1/4})$.

After the scaling of the magnetization $\alpha \to N^{-1/4}\alpha$, the distribution takes the form const × $\exp(-\alpha^4/12)$. It is interesting to note that this distribution, obtained as the stationary state of our evolution law, is precisely the ferromagnetic equilibrium distribution considered by Simon and Griffith[(16)] for the critical temperature $\beta J = 1$.

4. CONCLUDING REMARKS

Although we deal with a less complex situation, our study can be compared to the recent work by Hepp and Lieb[(17,18)] on the reservoir-driven laser. In both cases one derives an irreversible behavior (with a bifurcation in the motion) from a definite microscopic Hamiltonian model without the recourse to ad hoc statistical assumptions. We wish to point out some similarities and some differences.

In the case of the laser, initial states of the reservoirs may be taken at zero temperature and the laser makes its phase transition out of equilibrium when the coupling parameters to the reservoir are varied. Here the bifurcation occurs when the temperature of the initial state of the bath changes, the coupling being kept always the same.

The Markovian character of the evolution of the laser is obtained by the use of so-called "singular reservoirs." Singular reservoirs have a linear energy spectrum (hence not bounded below). They give rise to white noise processes where temporal correlations are Dirac functions. In our case, the Markovian character results from the application of the weak coupling limit. It does not seem that we could achieve the same description by the use of singular reservoirs, since white noise correlations are incompatible with the Kubo–Martin–Schwinger condition (4). As we have seen, this condition plays an essential role in bringing the temperature parameter into the equations of motion.

Another technical difference is that Hepp and Lieb focus their attention on the Heisenberg equations of motion of the total system, whereas we follow the evolution of states and probability distributions. Notice that in both cases, the passage from microscopic to macroscopic is done in the same way (they use the term classical to denote states that we call macroscopic).

The two models share common features essential for their solvability. The first is their mean field nature. Second, reservoirs are made of independent components attached to each degree of freedom of the system of interest (each individual atom or lattice site has its own private bath!). If all the spins were to interact with a common bath, correlation could be introduced between them via the bath in the course of the evolution. In particular, the assertion (a) of Proposition 2 would fail to be true. How to treat this point (not to mention dealing with more realistic non-quasifree baths) and how to go beyond mean field Hamiltonians are open problems in both models.

APPENDIX A

We recall briefly the framework of generalized master equations and the weak coupling limit following Ref. 11 (for a full mathematical analysis, see Ref. 11). Let Σ_1 and Σ_2 be two quantum systems described in Hilbert spaces $\mathscr{H}_1$ and $\mathscr{H}_2$, and let $\mathscr{B}_1$, $\mathscr{B}_2$, and $\mathscr{B}$ be the Banach spaces of trace class operators on $\mathscr{H}_1, \mathscr{H}_2$, and $\mathscr{H}_1 \otimes \mathscr{H}_2$, respectively. We assume that $\mathscr{H}_1$ is finite dimensional.

A self-adjoint Hamiltonian is given for the total system in the form $H = H_1 + H_2 + \lambda V$, where H_1 and H_2 govern the evolutions of Σ_1 and Σ_2, and the interaction V is assumed to be a bounded operator on $\mathscr{H}_1 \otimes \mathscr{H}_2$. These Hamiltonians induce evolution groups of isometries on $\mathscr{B}$ by the formulas

$$\begin{aligned} U_t^0\rho &= \exp[-i(H_1 + H_2)t]\, \rho \exp[i(H_1 + H_2)t] \\ U_t\rho &= \exp(-iHt)\, \rho \exp(iHt), \qquad \rho \in \mathscr{B} \end{aligned} \tag{A.1}$$

We suppose that the one-parameter group U_t^0 of free evolution is strongly continuous on $\mathscr{B}$ (i.e., with respect to the trace-norm topology) and hence is generated by a densely defined and closed operator A_0. We write

$$U_t^0 = \exp(A_0 t) \tag{A.2a}$$

with A_0 acting formally as the commutator

$$A_0\rho = -i[H_1 + H_2, \rho] \quad \text{on} \quad \mathscr{B} \tag{A.2b}$$

In the same way $U_t = \exp(At)$ is generated by $A = A_0 + \lambda\mathscr{V}$ with $\mathscr{V}\rho = -i[V, \rho]$, $\mathscr{V}$ being bounded on $\mathscr{B}$ (since by assumption V is bounded on $\mathscr{H}_1 \otimes \mathscr{H}_2$).

We identify now $\mathscr{B}_1$, the set of states of a system of interest, with a subspace of $\mathscr{B}$ by the projection

$$P\rho = (\mathrm{Tr}_{\mathscr{H}_2}\, \rho) \otimes \rho_2 \tag{A.3}$$

where ρ_2 is a given fixed state in $\mathscr{B}_2$. (The projector technique initiated by Zwanzig[19] has been used by many authors in various contexts; see references in Ref. 20.)

The partial trace operation $\mathrm{Tr}_{\mathscr{H}_2}$ on ρ maps $\mathscr{B}$ onto $\mathscr{B}_1$, whereas tensor multiplication by ρ_2 defines P as an action from $\mathscr{B}$ into $\mathscr{B}$.

The state $\rho_t = PU_t\rho_0 = (\mathrm{Tr}_{\mathscr{H}_2} U_t\rho_0) \otimes \rho_2$ contains all the information relevant to Σ_1 at time t when the initial condition for the coupled system is ρ_0.

We introduce three physical assumptions which are suitable when Σ_2 plays the role of a thermal bath.

(i) The two systems at time $t = 0$ are uncorrelated, that is, ρ_0 is of the form $\rho_1 \otimes \rho_2$.

(ii) The initial state ρ_2 of Σ_2 is stationary:

$$\exp(-iH_2t)\, \rho_2 \exp(iH_2t) = \rho_2$$

(iii) The average value of the interaction in the initial state of Σ_2 is zero:

$$\mathrm{Tr}_{\mathscr{H}_2} V\rho_2 = 0$$

The interpretation of (i) and (ii) is immediate and (iii) means that Σ_2 does not exert any external driving force on Σ_1.

It is convenient to choose the fixed state ρ_2 entering the definition (A.3) of P to be precisely the initial state of Σ_2. With this choice, one checks that, in virtue of (i)–(iii), P enjoys the following properties:

(i) The initial state is invariant under P:

$$P\rho_1 \otimes \rho_2 = \rho_1 \otimes \rho_2$$

(ii) The free evolution leaves the subspace $P\mathscr{B}$ invariant:

$$[P, U_t^0] = 0 \quad \text{on} \quad \mathscr{B}$$

(iii) The restriction of $\mathscr{V}$ to the subspace $P\mathscr{B}$ vanishes:

$$P\mathscr{V}P = 0$$

The state ρ_t, whose time development we want to investigate, can be written as well with (i) as

$$\rho_t = PU_t\rho_0 = PU_tP\rho_0$$

and the problem amounts to computing the restriction PU_tP of the full evolution to the subspace $P\mathscr{B}$. One finds that ρ_t satisfies the following integral equation:

$$\rho_t = U_t^0\rho_0 + \lambda^2 \int_0^t ds \int_0^s du U_{t-s}^0 K(\lambda, s - u)\rho_u \tag{A.4}$$

with kernel $K(\lambda, s)$ acting on $P\mathscr{B}$ given by

$$K(\lambda, s) = P\mathscr{V}(1 - P)U_s^\lambda(1 - P)\mathscr{V}P \tag{A.5}$$

where U_s^λ is the one-parameter group generated by $A_0 + \lambda(1 - P)\mathscr{V}(1 - P)$. Equation (A.4) is particularly well adapted to the study of time evolution of the subset $\mathscr{A}$ of the observables of Σ_1, which are constants of the motion of Σ_1,

$$\mathscr{A} = \{C \text{ on } \mathscr{H}_1: \quad [C, H_1] = 0\}$$

Clearly such observables will evolve at a slow rate when a weak coupling with an external system is switched on, and we shall restrict our attention to the time development of these observables.

Let us now describe the mechanism of the weak coupling limit on (A.4) for the class of observables in $\mathscr{A}$ (for a precise mathematical statement see Theorem 2.1 in Ref. 11).

If we evaluate both members of (A.4) on an observable C, belonging to $\mathscr{A}$, we get, with the notation $\operatorname{Tr} \rho_t C = \langle \rho_t, C\rangle$,

$$\langle \rho_t, C\rangle = \langle \rho_0, C\rangle + \lambda^2 \int_0^t ds \int_0^s du\, \langle K(\lambda, s - u)\rho_u, C\rangle$$

or in differential form

$$(d/dt)\langle \rho_t, C\rangle = \lambda^2 \int_0^t du\, \langle K(\lambda, u)\rho_{t-u}, C\rangle \tag{A.6}$$

The right-hand side of (A.6) is clearly a memory term involving ρ_s at all times s anterior to t. If we let $\lambda \to 0$ in (A.6) with t fixed, we come back to the noninteracting situation. However, we obtain a nontrivial effect if we simultaneously scale the observation time of Σ_1, setting $t = \lambda^{-2}\tau$ with τ fixed. We denote by $\mu_\tau = \rho_t$ the state expressed as a function of the new parameter τ and with this change of variable (A.6) becomes

$$(d/d\tau)\langle \mu_\tau, C\rangle = \int_0^{\lambda^{-2}\tau} du\, \langle K(\lambda, u)\mu_{\tau-\lambda^2 u}, C\rangle \tag{A.7}$$

Letting now formally $\lambda \to 0$ in (A.7), the limit state obeys an ordinary differential equation of semigroup type in the scaled time variable

$$(d/d\tau)\langle \mu_\tau, C\rangle = \langle G\mu_\tau, C\rangle \tag{A.8}$$

with

$$G = \int_0^\infty du\, K(0, u) \tag{A.9}$$

APPENDIX B

We give a functional analysis treatment of Eq. (11) with the purpose of exhibiting its structure when there are several stable equilibrium points. Let C_0 be the set of continuous functions with compact support on R and norm $|f| = \sup_{\alpha \in R} |f(\alpha)|$. Its dual space $\mathscr{B}$ is the Banach space of bounded measures on R with $|\mu| = \sup_{f \in C_0} [|\mu(f)|/|f|]$ = total variation of μ.

We shall also consider the subspace $\mathscr{B}_1$ of $\mathscr{B}$ of measures $d\mu(\alpha) = p(\alpha)\, d\alpha$ having densities $p(\alpha)$ with respect to Lebesgue measure with $|\mu| = \int |p(\alpha)|\, d(\alpha) < \infty$. The solution of (11) defines a semigroup V_τ on $\mathscr{B}$ by $(V_\tau \mu)(f) = \mu(f_\tau)$. When $T < T_c$ there are two stationary measures $\delta_{m_\pm}$, $V_\tau \delta_{m_\pm} = \delta_{m_\pm}$, which are the stable equilibrium points corresponding to the two possible thermodynamic phases. We shall assume in the following that $\gamma(x)$ does not vanish and is twice continuously differentiable. We introduce the semigroups of linearized motion around the equilibrium point $m_\pm$,

$$(U_\tau \mu)(f) = \mu(f_\tau^{\pm}), \qquad f_\tau^{\pm}(\alpha) = f((\alpha - m_\pm)\exp(-c\tau) + m_\pm)$$

with c given by (14).

We are interested in the following questions:

1. In which sense does the linearized motion approximate the full motion?

2. How does one determine in general the sets of initial conditions in $\mathscr{B}$ that are attracted by stable equilibrium points?

Lemma 1. V_τ and $U_\tau^{\pm}$ are bounded groups of isometries on $\mathscr{B}$, $|V_\tau \mu| = |U_\tau^{\pm} \mu| = |\mu|$.

V_τ is a group since (12) can also be solved for $\tau < 0$. $|(V_\tau \mu)(f)| \leqslant |\mu|\, |f_\tau| \leqslant |\mu|\, |f|$ implies $|V_\tau \mu| \leqslant |\mu|$ and since this holds for all τ and all μ, $|V_\tau \mu| = |\mu|$.

Proposition 4.[8]

$$w\text{-}\lim_{\tau \to \infty} U^{\pm}_{-\tau} V_\tau \mu \qquad \text{exists for all } \mu \text{ in } \mathscr{B} \tag{B.1}$$

$$s\text{-}\lim_{\tau \to \infty} V_{-\tau} U_\tau^{\pm} \mu \qquad \text{exists for all } \mu \text{ in } \mathscr{B}_1 \tag{B.2}$$

Proof. Let μ be in $\mathscr{B}$ and f a continuously differentiable function in C_0. We show that

$$|(U^{\pm}_{-\tau_1} V_{\tau_1} \mu)(f) - (U^{\pm}_{-\tau_2} V_{\tau_2} \mu)(f)| \leqslant \int_{\tau_2}^{\tau_1} \left| \frac{d}{d\sigma} (U^{\pm}_{-\sigma} V_\sigma \mu)(f) \right| d\sigma \tag{B.3}$$

is a Cauchy sequence in τ_1 and τ_2. One has

$$\frac{d}{d\sigma}(U^{\pm}_{-\sigma} V_\sigma \mu)(f) = \frac{d}{d\sigma}(V_\sigma \mu)(f^{\pm}_{-\sigma}) = (V_\sigma \mu)\left[h(f^{\pm}_{-\sigma})' + \frac{d}{d\sigma} f^{\pm}_{-\sigma} \right]$$

$$\leqslant |\mu| \left| h(f^{\pm}_{-\sigma})' + \frac{d}{d\sigma} f^{\pm}_{-\sigma} \right|$$

[8] $w - \lim_{n \to \infty} \mu_n = \mu$ means $\lim_{n \to \infty} \mu_n(f) = \mu(f)$ for all $f \in C_0$, and $s - \lim_{n \to \infty} \mu_n = \mu$ means $\lim_{n \to \infty} |\mu_n - \mu| = 0$.

We calculate

$$\sup_\alpha \left| \left[h(\alpha)\frac{\partial}{\partial\alpha} + \frac{\partial}{\partial\sigma}\right] f((\alpha - m_+)\exp(c\sigma) + m_+) \right|$$
$$= \sup_\alpha |[h(\alpha) + c(\alpha - m_+)]\exp(c\sigma) f'((\alpha - m_+)\exp(c\sigma) + m_+)|$$
$$= \sup_\alpha |[h((\alpha - m_+)\exp(-c\sigma) + m_+)\exp(c\sigma) + c(\alpha - m_+)] f'(\alpha)|$$
$$= \exp(-c\sigma)\sup_\alpha \left| h''(\xi)\frac{(\alpha - m_+)^2}{2} f'(\alpha)\right| = \exp(-c\sigma) M_f \tag{B.4}$$

where we have used the Taylor formula:

$$h((\alpha - m_+)\exp(-c\sigma) + m_+) = -c(\alpha - m_+)\exp(-c\sigma) + \frac{\frac{1}{2}(\alpha - m_+)^2}{2} h''(\xi)\exp(-2c\sigma)$$
$$\min(\alpha, m_+) \leqslant \xi \leqslant \max(\alpha, m_+)$$

Therefore $|(d/d\sigma)(U_{-\sigma}^{\pm} V_\sigma \mu)(f)| \leqslant \exp(-c\sigma)|\mu| M_f$ is integrable at $\sigma = \infty$, showing that (B.3) converges as $\tau_1, \tau_2 \to \infty$. Since the set of differentiable functions is dense in C_0 and the groups V_τ and U_τ^{+} are bounded uniformly in τ, the limit (B.1) exists for all f in C_0. This proves the first assertion of the proposition.

For the second one, we proceed in a similar way. Let μ in $\mathscr{B}_1$ have a continuously differentiable density $p(\alpha)$ with compact support; then

$$|V_{-\tau_1} U_{\tau_1}^{+}\mu - V_{-\tau_2} U_{\tau_2}^{+}\mu| \leqslant \int_{\tau_2}^{\tau_1} |(d/d\sigma) V_{-\sigma} U_\sigma^{+}\mu|\, d\sigma$$

From the definitions and with the help of an integration by parts we get the estimate as in (B.4):

$$\left| \frac{d}{d\sigma}(V_{-\sigma} U_\sigma^{+}\mu)(f)\right|$$
$$\leqslant |f| \int \left| \frac{d}{d\alpha}\{p(\alpha)[h((\alpha - m_+)\exp(-c\sigma) + m_+)\exp(c\sigma) + c(\alpha - m_+)]\}\right| d\alpha$$
$$\leqslant \exp(-c\sigma)|f| M_\mu$$

Hence the limit (B.2) exists, and on all of $\mathscr{B}_1$ by density.

We denote by $\Gamma_\pm\mu$ for μ in $\mathscr{B}$ and $\Omega_\pm\mu$ for μ in $\mathscr{B}_1$ the linear applications defined by the limits (B.1) and (B.2), respectively. Then $\Gamma_\pm$ and $\Omega_\pm$

enjoy the following properties, which one derives immediately from the definitions and from Proposition 4.

Lemma 2. (i) $|\Gamma_{\pm}\mu| \leqslant |\mu|$, $\mu \in \mathscr{B}$; $|\Omega_{\pm}\mu| = |\mu|$, $\mu \in \mathscr{B}_1$.
(ii) $V_\tau\Omega_{\pm}\mu = \Omega_{\pm} U_\tau{}^{\pm}\mu$, $\mu \in \mathscr{B}_1$.
(iii) $\Gamma_{\pm}\Omega_{\pm}\mu = \mu$, $\mu \in \mathscr{B}_1$.
(iv) $\Gamma_{\mp}\Omega_{\pm}\mu = 0$, $\mu \in \mathscr{B}_1$.

Property (iv) holds because

$$\Gamma_{\mp}\Omega_{\pm}\mu = w\text{-}\lim_{\tau\to\infty} U^{\mp}_{-\tau}V_\tau V_{-\tau}U_\tau{}^{\pm}\mu = w\text{-}\lim_{\tau\to\infty} U^{\mp}_{-\tau}U_\tau{}^{\pm}\mu$$

and by dominated convergence,

$$(U^{\mp}_{-\tau}U_\tau{}^{\pm}\mu)(f) = \int d\mu(\alpha) f(\alpha + (m_{\pm} - m_{\mp})[\exp(c\tau) - 1])$$

tends to zero for $f \in C_0$ as $\tau \to \infty$.

To interpret these results, we observe that if μ belongs to the range of Ω_+ (Ω_-), $V_\tau\mu$ behaves asymptotically as the linearized evolution $U_\tau{}^+\Gamma_+\mu$ ($U_\tau{}^-\Gamma_-\mu$). Indeed, μ is of the form $\mu = \Omega_+\nu$ for some $\nu \in \mathscr{B}_1$ and with (B.2) and (iii) of Lemma 2,

$$\begin{aligned} |V_\tau\mu - U_\tau{}^+\Gamma_+\mu| &= |(V_\tau\Omega_+ - U_\tau{}^+)\nu| \\ &= |(V_{-\tau}U_\tau{}^+ - \Omega_+)\nu| \to 0, \qquad \tau\to\infty \end{aligned}$$

This implies in particular that the range of Ω_+ [Ω_-] belongs to the domain of attraction of the equilibrium point δ_{m_+} [δ_{m_-}]. These two sets are disjoint by property (iv) of Lemma 2, and one checks easily that measures in the range of Ω_+ [Ω_-] have support in $(0, \infty)$ [$(-\infty, 0)$], as we have seen in Section 3, case 1, $T < T_c$.

It is interesting to observe that this mathematical structure is analogous to that which is encountered in multichannel scattering processes. The different thermodynamic phases are the channels and the linearized evolutions $U_\tau{}^{\pm}$ are the free channel evolutions. Conditions (B.1) and (B.2) are the asymptotic conditions with channel "wave operators" $\Omega_{\pm}$. The ranges of Ω_+ and Ω_- (channel subspaces) correspond to the domains of attraction of stable equilibrium points.

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